The Intrinsic Scale Of Networks Is Small

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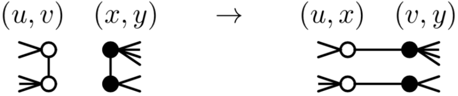
At what (small) scale does a network identify itself?

- "From what?", one asks. "From another network", another responds.
- The intrinsic scale of one network should not depend on the properties of another network.
- We propose a methodology, which, given a network N with n vertices and m edges, extracts the intrinsic scale.
- Results from several networks reveal a surprising conclusion:

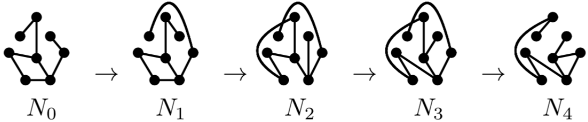
  The intrinsic scale of real networks is $7 - 20$ *vertices*
- This means, these networks have non-trivial structure at small scales.
- Old school parameters such as power-law exponents and clustering coefficients may not be stable at this scale.

Subgraphs Induced by Random Walks

- Network N has structure at scale κ if typical size-κ subgraphs from N are distinguishable from size-κ subgraph in a randomized copy of N.
- If N_δ is a perturbed copy of N where δ is the extent of perturbation, then $N_0 = N$ and N_∞ is a random graph with the same degrees as N
- To construct N_δ, we use random edge-swaps to rewire the network as shown below.



- Observe that an edge swap preserves every vertex-degree. This is what a sequence of edge swaps on a toy graph looks like:



Defining the Intrinsic Scale

- Let a random walk W traverse a random incident edge at each step. After visiting κ different nodes, construct the induced subgraph:

$$W : (N_\delta, \kappa) \mapsto G(\kappa, \delta)$$

  where $G(\kappa, \delta)$ is a random graph that depends on N_δ, the starting node and the edges traversed.
- The process W induces a distribution $p_{\kappa,\delta}$ over graphs with κ vertices.
- Existing structure in N_0 at scale κ is lost during δ steps of randomization if $p_{\kappa,0}$ and $p_{\kappa,\delta}$ are distinguishable.
- The Bayes *optimal* classifier for the distributions $p_{\kappa,0}$ and $p_{\kappa,\delta}$ has classification accuracy:

$$\Delta(\kappa, \delta) = \frac{1}{2} \sum_G \max\{p_{\kappa,0}(G), p_{\kappa,\delta}(G)\}$$

Defining the Intrinsic Scale

- For $\tau > \frac{1}{2}$, let $\kappa^*(\tau, \delta)$ be the minimum scale κ at which one can distinguish κ-sized subgraphs of N from those in N_δ with accuracy at least τ,

  $$\kappa^*(\tau, \delta) = \text{min}\{\kappa | \Delta(\kappa, \delta) \geq \tau\}.$$

  The intrinsic scale is

  $$\kappa^*(\tau) = \lim_{\delta \to \infty} \kappa^*(\tau, \delta).$$

- The process W is responsible for producing κ-sized subgraphs, the details of which can affect specific values of κ^*.

Algorithm: Estimating the Bayes Optimal Classifier

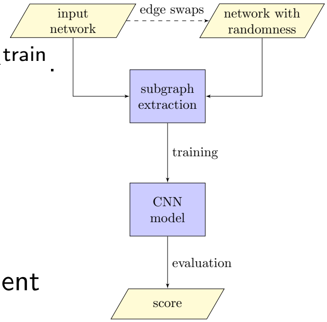
Estimate $\Delta(\kappa, \delta)$:

1. Given N_0, construct N_δ using δ edge-swaps.
2. Sample κ-sized subgraphs from N_0 and N_δ to get a training set $\{G_{\kappa,0}\}^{\text{train}}$ and $\{G_{\kappa,\delta}\}^{\text{train}}$.
3. Use the training set to learn a classifier

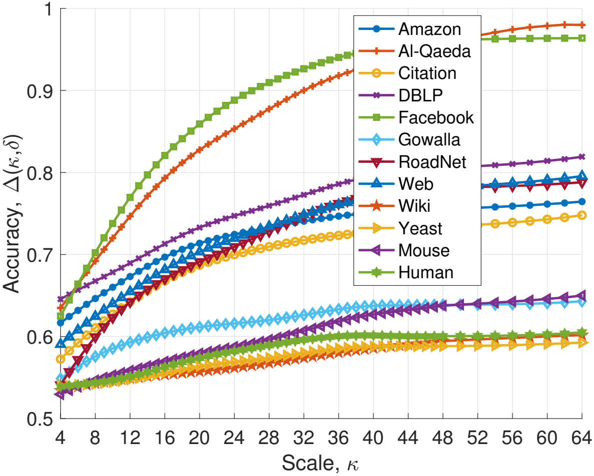
$$g_{\kappa,\delta} : G_\kappa \mapsto \pm 1.$$

   ($+1$ for N_0, -1 for N_δ).

4. Test the learned classifier $g_{\kappa,\delta}$ on independent test subgraphs $\{G_{\kappa,0}\}^{\text{test}}$ and $\{G_{\kappa,\delta}\}^{\text{test}}$.
5. Return $\hat{\Delta}(\kappa, \delta)$, the test accuracy of $g_{\kappa,\delta}$.



Robust Networks are Mostly Immune to Small Perturbations



Small δ:Robustness; Large δ:Structure

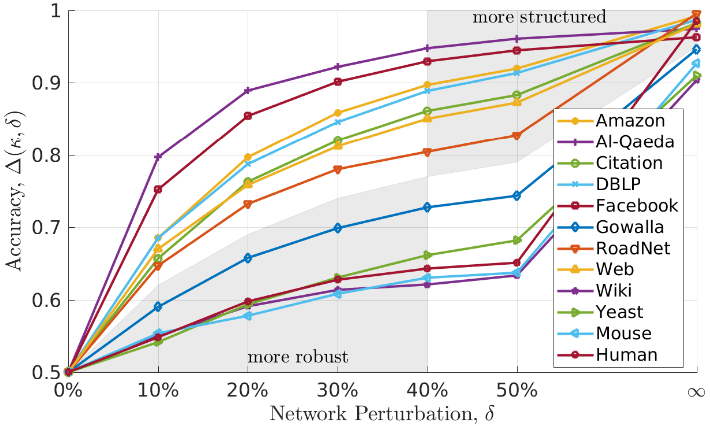


Figure: Learning curves for $\kappa = 16$

We are an Outlier

Summary of findings:

- Trend: High $\gamma \implies$ High $\kappa*$
- Exception:
  Human PPI is very robust and yet very structured.
- The intrinsic scale of real networks is between $7 - 20$

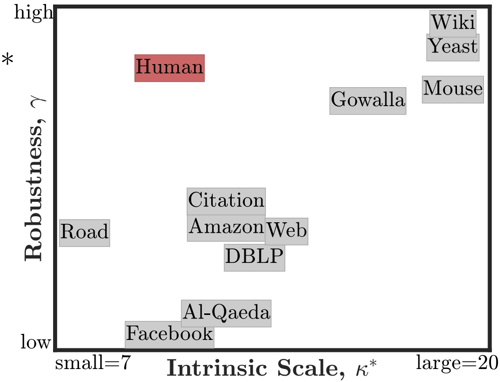


Figure: 2D Landscape of γ and $\kappa*$

Future Directions

What next?

- Can we construct networks and models wrt. the structure-robustness trade-off?
- How to construct networks that break said trade-off?
- Can we use the intrinsic scale to improve other network analysis algorithms like clustering?

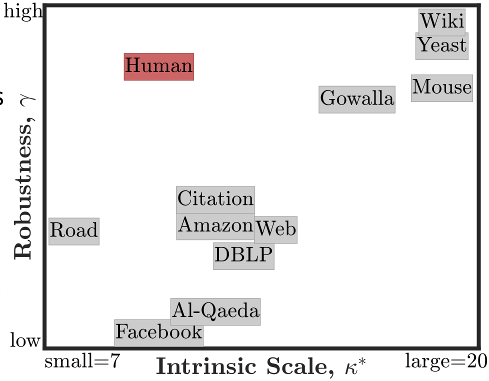


Figure: 2D Landscape of γ and $\kappa*$

Questions?

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